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PRACTICAL INFORMATION

First home assignment:

- paper and web version today, due next Thursday (October 8),
- you **must** consult the “Instructions for home assignments” page, for rules and frequently asked questions,
- worth 15% of total course mark, and covers only material from Sessions 1–4.

Today’s lecture:

- more about **random variables**:¹
 - * **distribution of the sample mean** (and sample proportion²),
 - * some “great” (mathematical) results:
law of large numbers, and **central limit theorem**,
- **statistical inference**,
 - * continuing with **estimation**,
 - * **confidence intervals**,³
 - **tests** (significance, *P*-value),⁴

but a critical discussion of statistical inference will wait until Session 12.

¹ PSLS 4e: Chapter 13; S: Chapter 6; IPS 7e: Sections 3.3+5.1-2.

² The approximation of a binomial distribution by a normal distribution (Appendix) is not part of the course syllabus.

³ PSLS 4e: Chapter 14-15 (parts); S: Chapter 7; IPS 7e: Section 6.1.

⁴ PSLS 4e: Chapter 14-15 (parts); S: Chapter 8; IPS 7e: Sections 6.2-3.

BIAS AND VARIABILITY

Model of our data X_1, \dots, X_n involves a parameter θ (in our examples, the mean μ or the proportion p).

Definition: an estimate $\hat{\theta}$ of a parameter θ is **unbiased**, if:

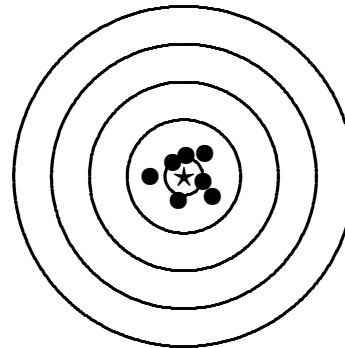
$$E \hat{\theta} = \theta,$$

i.e., *on the average*, the estimate “hits right at θ ”.⁵

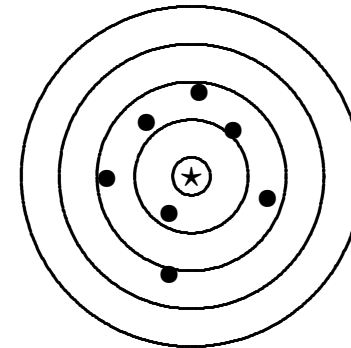
Targeting analog:

($\star \sim$ true value,
 $\bullet \sim$ observed values
 in replications of
 the experiment)

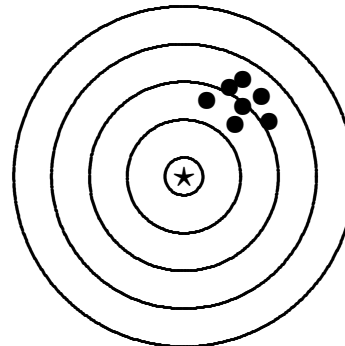
low variability, low bias



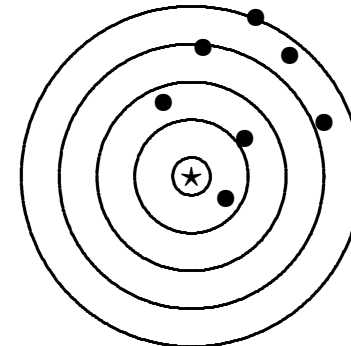
high variability, low bias



low variability, high bias



high variability, high bias



Challenge: how would the corresponding sampling distributions (of $\hat{\theta}$) look?

⁵ Generally (in statistics), the **bias** of an estimate is: $\text{bias}(\hat{\theta}) = E \hat{\theta} - \theta$.

STATISTICAL PROPERTIES OF SAMPLE STATISTICS

Terminology (mine!): i.i.d. variables $X_1, \dots, X_n \sim$ **independent** and with **same distribution**.

Result: For i.i.d. variables X_1, \dots, X_n with mean μ and standard deviation σ , we have

$$E\bar{X} = \mu, \quad \text{Var}\bar{X} = \sigma^2/n, \quad \text{and} \quad \text{SE} = \text{sd}\bar{X} = \sigma/\sqrt{n}.$$

One important implication hereof is that the estimate

$$\hat{\mu} = \bar{X} \quad \text{is **unbiased** for } \mu.$$

Summary of estimation from a single sample:

- for **estimation of a mean** we use

$$\hat{\mu} = \bar{X} \text{ — unbiased with } \text{sd}(\hat{\mu}) = \sigma/\sqrt{n}.$$

- for **estimation of a proportion** (observing X out of n)⁶

$$\hat{p} = X/n = \bar{S} \text{ — unbiased with } \text{sd}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}.$$

Furthermore, if the variables X_1, \dots, X_n are independent and **normally distributed** $N(\mu, \sigma)$, then

$$\bar{X} \sim N(\mu, \sigma/\sqrt{n}).$$

Note: the **new result** here is that \bar{X} is **normally distributed**, actually any linear combination of independent normal variables⁷ is again normally distributed.

⁶ We have $X = S_1 + \dots + S_n$, where the S_i are 1 (\sim event) or 0 (\sim non-event); the S_i are called indicators of the events, or binary or **Bernoulli** variables.

⁷ For example, $Y = a_1X_1 + a_2X_2 + a_3X_3$, where a_1, a_2 and a_3 are numbers.

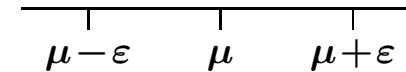
LAW OF LARGE NUMBERS (LLN)

= **mathematical result** (probability theory):

If X_1, \dots, X_n are i.i.d. variables with mean μ , then:

$$P(\mu - \varepsilon \leq \bar{X} \leq \mu + \varepsilon) \rightarrow 1$$

as $n \rightarrow \infty$, for any $\varepsilon > 0$.



Less formally, for “large” n :

- $(X_1 + \dots + X_n)/n = \bar{X} \approx \mu$,
- eventually (when n is large enough): $\mu - \varepsilon \leq \bar{X} \leq \mu + \varepsilon$ with very high probability, for any $\varepsilon > 0$.⁸

Illustrations of LLN:

- PSLS applet “Law of Large Numbers”,
- PSLS applet “Probability” (for binary outcome) you have tried already.⁹

Implications of LLN:

- “stabilizing behavior” of a series of averages (or proportions) that we have seen previously (simulation).
- strong (good!) property of the sample mean as an estimate of the population mean.

⁸ Intuitively, the required n depends on both $\varepsilon > 0$ and the targeted probability.

⁹ As mentioned on the previous slide, the sample proportion is indeed a sample mean (for the indicators S_i).

CENTRAL LIMIT THEOREM (CLT)

= **mathematical result** (probability theory):

If X_1, \dots, X_n are i.i.d. variables with mean μ and stand. dev. σ , then the cumulative probabilities (“to the left”) for the standardized sum satisfy:

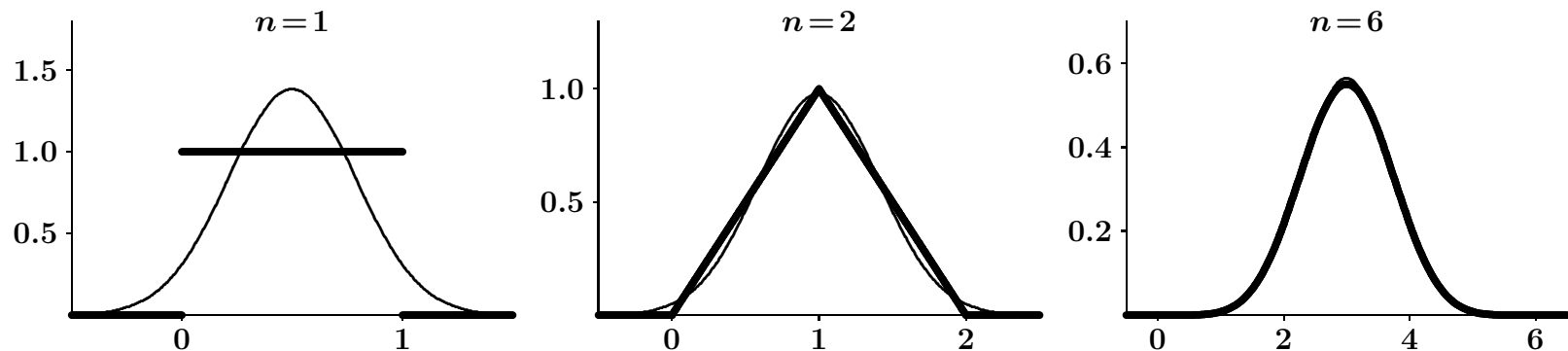
$$P\left(\frac{X_1 + \dots + X_n - n\mu}{\sqrt{n}\sigma} \leq x\right) \rightarrow P(Z \leq x) \quad \text{as } n \rightarrow \infty,$$

for any real number x , and where (as usual) $Z \sim N(0, 1)$.

Less formally, for “large” n :

$$\begin{aligned} X_1 + \dots + X_n &\approx N(n\mu, \sqrt{n}\sigma), \\ (X_1 + \dots + X_n)/n = \bar{X} &\approx N(\mu, \sigma/\sqrt{n}). \end{aligned}$$

Illustration — approximation of a sum of uniform distributions: (bold=exact density, thin=normal approximation):¹⁰



¹⁰ See also the PSLS Central Limit Theorem applet.

IMPLICATIONS OF THE CLT

Remarks on CLT (central limit theorem):

- CLT deals with **i.i.d. variables**, both assumptions involved are crucial,
- we know already that a sum/average of normal random variables is (exactly) normal, but the CLT says that **any sum/average of i.i.d. variables** is approx. normal,
- **intuitively** surprising: any skewnesses or irregularities in distribution smoothed out (by sum/average),
- implies a **special role** of the normal distribution,
- partial **justification** for general use of the normal distribution (some outcomes may be thought of as an addition of many small effects, e.g. growth, yield),
- **stronger** result than **LLN** (the law of large numbers, telling us that the distribution of \bar{X} narrows in around μ), because distribution is known as well.

Applications:

- to sum of binary/Bernoulli variables (= binomial variable)
⇒ approximation of binomial distribution by normal distribution (Appendix),
- generally to **average of i.i.d. variables** ⇒ approximate statistical inference for sample average \bar{X} without assuming any particular distribution for X s.

INTRODUCTION TO CONFIDENCE INTERVALS

Data example: 10 calves on infected pasture, parasite egg counts X_1, \dots, X_{10} .

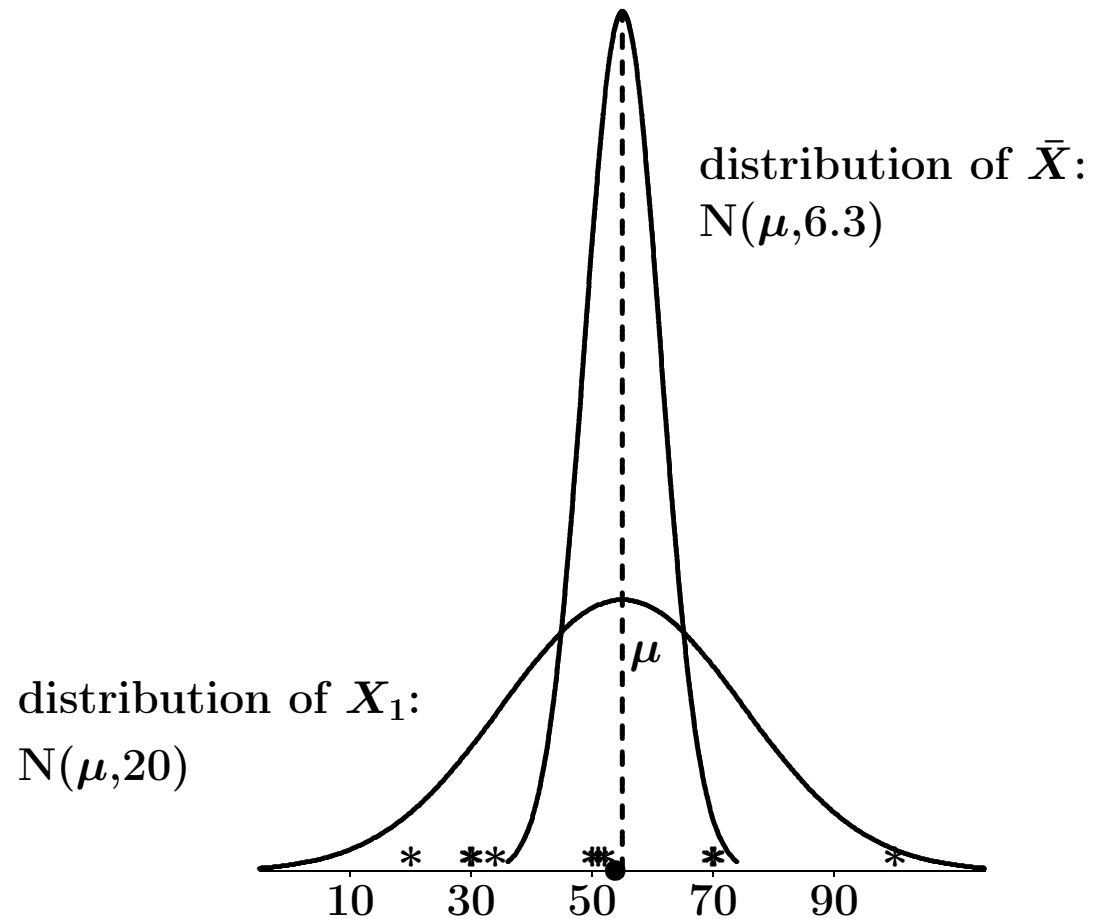
- **Model:** X_1, \dots, X_{10} i.i.d. variables with mean μ .
- **Estimate:** $\hat{\mu} = \bar{X} = 51.2$.

What does this tell us about μ ?

- * almost nothing,¹¹
- * $\mu = 51.2$,
- * μ is close to 51.2
(say, within ± 1),
- * μ is somewhere around 51.2
(say, within ± 15).

Same question,

if we know that $\sigma_{\bar{X}} \approx 6.3$?
($\sigma = 20$, and $\sigma_{\bar{X}} = \sigma / \sqrt{10}$)



¹¹ Estimates without any indication of precision are not worth much.

A REAL-LIFE CONFIDENCE INTERVAL

From the news (September 2004):

A poll by the Centre for Research and Information on Canada shows that 61% of Canadians believe that religious practice is an important factor in the moral and ethical lives of Canadians. [...]

The poll of 1500 adult Canadians was conducted June 16-21, 2004, and is considered accurate within plus or minus 2.5 per cent 19 times out of 20. [...]

(Province breakdown: Atlantic 76%, Quebec 44%, etc.; corresponding number in 1980: 79%)

In statistical terms:

- **estimates**: proportions of respondents indicating religious practice to be important factor (61%, 79%),
- **confidence intervals**: limits of $\pm 2.5\%$ with a **confidence** of 19 times out of 20 (95%):
 - * loosely stated, this means that there is 95% probability that the **true proportions** are within $\pm 2.5\%$ of the estimates,
 - * we'll make the precise meaning clear shortly.

Confidence limits aid in (are crucial for) the **interpretation** of the estimates; here they show the difference between now and 1980 is huge, as are the differences between provinces (although with larger intervals, why?).

CONFIDENCE INTERVAL (CI) BASICS

Idea(s) and Concepts:

- combine estimate $\hat{\mu}$ and its standard error SE (or, $\text{sd}(\hat{\mu})$)¹² to a statement about μ
⇒ interval estimate = **confidence interval**,
- rarely able to say something for certain about μ
⇒ need to set a level of certainty for our statement = **confidence level**,
- **confidence levels** are denoted by C , and are typically of the form $1 - \alpha$, where α is the **error level**,
- **mostly used values** are
 - $C = 0.90$ (90%), $\alpha = 0.10$ (10%),
 - $C = 0.95$ (95%; “19 times out of 20”), $\alpha = 0.05$ (5%),
 - $C = 0.99$ (99%), $\alpha = 0.01$ (1%),

with high (low) values of C corresponding to high (low) certainty (confidence),

- most confidence intervals are **symmetric** about the parameter estimate, that is, of the form

$$\mu : \hat{\mu} \pm \text{margin of error},$$

and the **margin of error** is very often calculated as

“percentile \times SE”.

¹² Recall that the standard error of an estimate is the standard deviation in its distribution.

CONFIDENCE INTERVAL FOR POPULATION MEAN

Formula: Let X_1, \dots, X_n be a SRS (i.i.d.) from a population with mean μ (unknown) and standard deviation σ (**known**). Then an (approximate) **95% confidence interval** for μ is

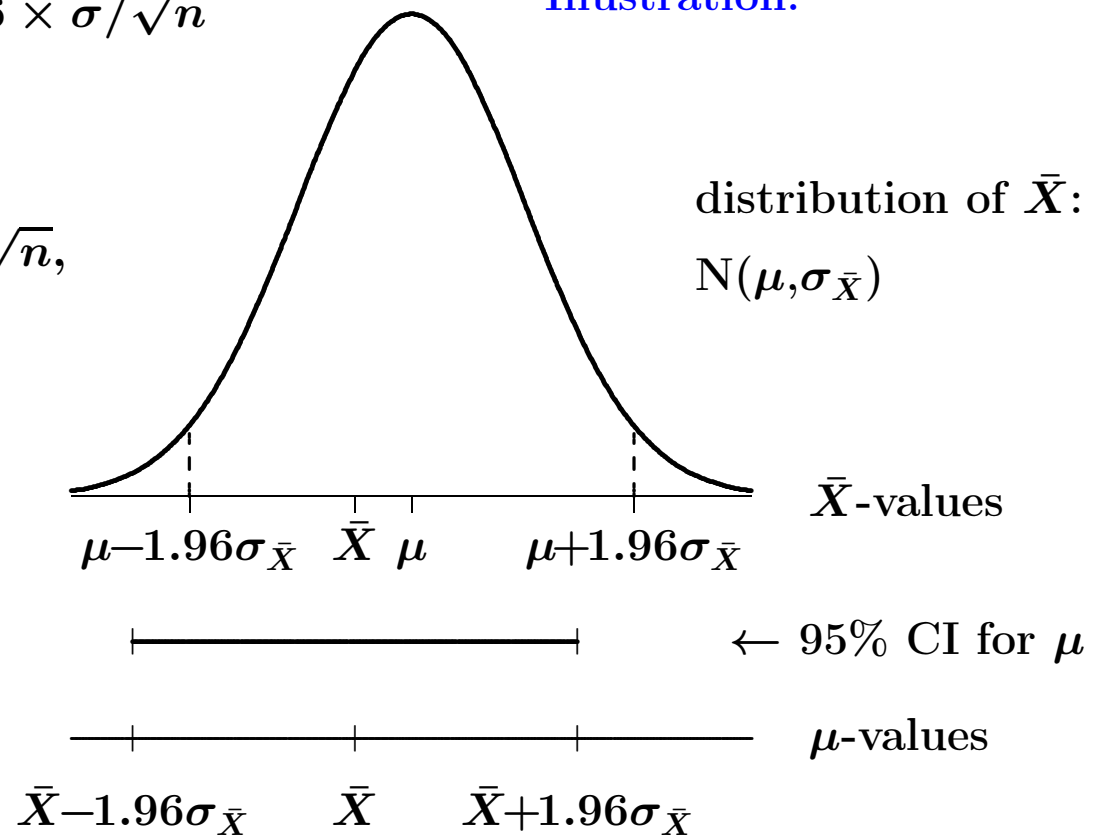
$$95\% \text{ CI for } \mu : \bar{X} \pm 1.96 \times \sigma / \sqrt{n}$$

Illustration:

Generally, an (approximate) **$(1 - \alpha)$ confidence interval** is

$$(1 - \alpha) \text{ CI for } \mu : \bar{X} \pm z^* \times \sigma / \sqrt{n},$$

where z^* is a suitable percentile¹³ in $N(0,1)$.



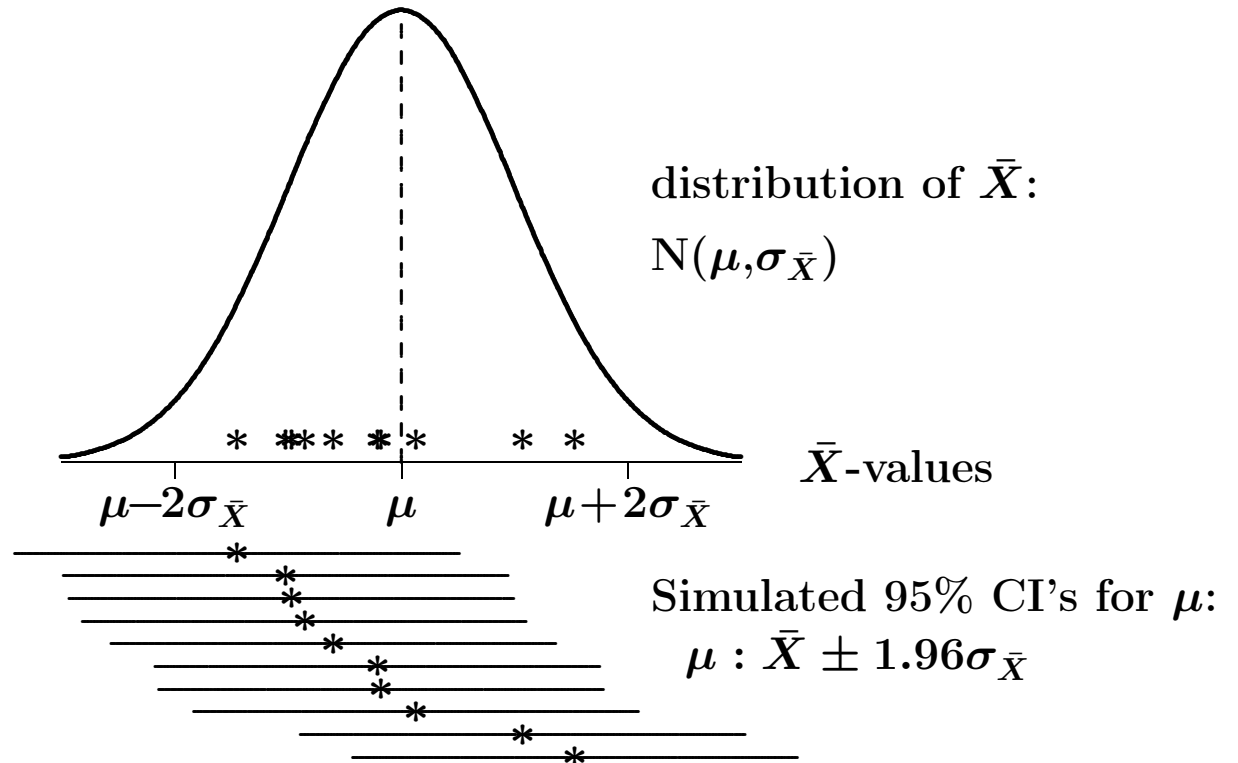
¹³ Formally, $z^* = z_{1-\alpha/2}$ is the $(1 - \frac{\alpha}{2})$ percentile in $N(0,1)$; z^* -values are found in PSLS: Table C, IPS: Table D, or as “critical values” in S: Table 3.

INTERPRETATION OF CONFIDENCE INTERVALS

Simulation of 10 sample means
from $N(\mu, \sigma_{\bar{X}})$:

Frequency interpretation
of confidence intervals:

- *on the average*,
95% of CI's will contain μ ,
- the **randomness** is
in the method,
not in μ (fixed value),
- for each specific interval,
either μ is in interval
or μ is outside interval (but we don't know which is true)...



Assumptions of confidence interval for population mean:

- **i.i.d.** sample (independent, identically distributed),¹⁴
- (approximate) **normal distribution** of \bar{X} ,
- **σ known** (in practice, rarely a reasonable assumption).

¹⁴ PSLS stresses the assumption of a simple random sample from the population.

EXERCISES 6.20, 6.47 AND 6.55

Exercise 6.20: Confidence intervals in opinion polls:

- (a) **No**; we can only become certain about the population value by sampling the entire population (not feasible). Every confidence interval has a confidence level $< 100\%$,
- (b) The interval (27% to 33%) was based on a **method** that includes the true population percentage 95% of the time (with repeated sampling),
- (c) For a 95% CI, $z^* = 1.96 \Rightarrow \sigma_{\text{estimate}} = 0.03/1.96 = 0.0153$,
- (d) **No**; it only accounts for random fluctuations.

Exercise 6.47: Null and alternative hypotheses for testing problems:

- (a) $H_0: \mu = 18$ and $H_a: \mu < 18$,
- (b) $H_0: \mu = 50$ and $H_a: \mu > 50$,
- (c) $H_0: \mu = 24$ and $H_a: \mu \neq 24$.

Exercise 6.55: P -values against one/two-sided alternatives (from observed $z = 1.8$):

- (a) $P = P(Z > 1.8) = 1 - 0.9641 = 0.0359 \sim$ **significant** at the 5% significance level,
- (b) $P = P(Z < 1.8) = 0.9641 \sim$ **non-significant** (at any meaningful significance level),
- (c) $P = 2 \times P(Z > 1.8) = 0.072 \sim$ **non-significant** at the 5% significance level.

TWO EXAMPLES OF TEST PROBLEMS

Example I: **Testing of taste** (example not in textbooks):

- **aim**: compare two brands of wine (beer, milk, cheese...),
- “**duo-trio test**” with one subject (person):
 - two anonymized samples, one of each brand,
 - third sample of known type,
 - subject may taste all 3 samples, as (s)he likes,
 - **task**: determine brands of two unknown samples,
- **repeating the experiment**, subject scores x out of n (e.g., 6 out of 8) correctly — how to determine if result has not occurred by chance (“luck”)?
- **statistical problem**, because of randomness associated with “guessing” (even if qualified guessing).

Example II: **Laboratory analysis** of active ingredient in specimens:

(Exercise 14.5 of PSLS 4e)

- **data** from 3 analyses of one specimen: 0.8403, 0.8363, 0.8447 (in g/l),
- **aim**: evaluate producer’s specified content of 0.86 g/l,
- **statistical problem**, because of random measurement errors in laboratory.

INTRODUCTION TO STATISTICAL TESTING

Consider the “duo-trio” testing problem, and let X denote the number of “successes” for one subject in 8 trials.

- **binomial setting** $\Rightarrow X \sim$ binomial distribution $B(8, p)$,
- **if guessing**, the probability p in each trial must be $p=0.5$ — state this as our **null hypothesis** $H_0: p=0.5$,
- **under H_0** : $X \sim B(8, 0.5)$:

x	0	1	2	3	4	5	6	7	8
$P(X=x)$	0.004	0.03	0.11	0.22	0.27	0.22	0.11	0.03	0.004
- **alternatively** to H_0 we must have $p > 0.5$ (unless subject messes up the experiment) — state this as our **alternative hypothesis** $H_a: p > 0.5$,

If subject gets **all trials right** ($X=8$):

- * **probability of event happened by chance**: $P = 0.004$,
- * by low P -value, we have little confidence in H_0 (because observed event unlikely to happen if H_0 was true) \Rightarrow **reject** H_0 and prefer H_a , (but H_0 could be true...),

If subject gets **6 out 8 trials right** ($X=6$):

- * **probability of actual event or *more extreme* events**:
 $P = P(X \geq 6) = P(X=6) + P(X=7) + P(X=8) = 0.14$,
- * by not too low P -value, observed $X=6$ does not seem unreasonable under H_0 (might have happened by chance) \Rightarrow **cannot reject** H_0 , (but H_0 could be false...).

COMPONENTS OF A STATISTICAL TEST

Statistical **Model** — main examples so far: $X \sim B(n, p)$, and X_1, \dots, X_n i.i.d. (SRS) of population (μ, σ) .

Statistical **Hypothesis**:

- **statement/assertion** about the model (parameter(s) of the model) which is either true or false,
- **null hypothesis** H_0 — the one investigated,
- **alternative hypothesis** H_a — the one to hold, if H_0 is not true.

Statistical **Test statistic** (or test variable):

- computed from the data (in some cases, the entire data),
- “measures” how well **the data correspond** to H_0 compared to H_a .

P-value (or significance probability):

- the probability, computed under H_0 (assuming H_0 is true), that the test statistic takes a value **as extreme as or more extreme than** (in the direction of H_a) the observed value from the data,¹⁵

¹⁵ The *P*-value expresses how surprising the observed outcome would be if H_0 was true.

- low P -values provide evidence against H_0
 \Rightarrow rejection of H_0 (and acceptance of H_a , **strong conclusion**),¹⁶
- high P -values provide no (convincing) evidence against H_0
 $\Rightarrow H_0$ cannot be rejected (**weak conclusion**).

Significance level α :

- **artificial** borderline/cut-off set for convenience between **significant** (i.e., $P \leq \alpha$) and **non-significant** (i.e., $P > \alpha$) results,¹⁷
- **by convention** set at 0.05, or less commonly at 0.10, 0.01, etc.

Analogs between reasoning in law and statistics:

Concept	Law	Statistics
initial position	innocence	null hypothesis
claim	guilt	alternative hypothesis
information	evidence	data
decision rule	guilt beyond	$P <$
to rule out chance	reasonable doubt	significance level
conclusive result	guilt	alternative hypothesis
inconclusive	innocence	null hypothesis
result	could not be ruled out	could not be rejected

¹⁶ “If the P -value is low, the null hypothesis must go.” (Keith Bower; media links)

¹⁷ No uniform rule exists for whether ($P = \alpha$) is considered significant or not.

TEST FOR POPULATION MEAN

Setting for test of population mean:

- **Model:** X_1, \dots, X_n i.i.d. from distribution (μ, σ) ,
 - * **assume** (approximate) normal distribution of \bar{X} ,
 - * **assume** σ known (in practice, rarely a reasonable assumption).
- **Null Hypothesis** H_0 : $\mu = \mu_0$, where μ_0 is a known, fixed value (very often, $\mu_0=0$),
- **Alternative Hypothesis** H_a : $\mu \neq \mu_0$,

- **z-test statistic** computed as:

$$z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1) \text{ under } H_0,$$

- **P-value** computed as:

$$P = 2 \times P(Z \geq |z|) = 2 \times P(Z \leq -|z|).$$

Example II: Laboratory analysis,

- **Data:** X_1, X_2, X_3 ; $n=3$, $\bar{X}=0.8404$, $\sigma=0.0068$ known,
- **Hypotheses:** $H_0: \mu = 0.86$, $H_a: \mu \neq 0.86$,
- **Test statistic:** $z = (0.8404 - 0.86)/(0.0068/\sqrt{3}) = -4.98$,
- **P-value:** $P = 2 \times P(Z \leq -4.98) < 2 \times 0.0002 = 0.0004$,
- **Conclusion:** reject H_0 and accept H_a ; **strong indication** that the specimen content is not as specified (i.e., **lower**, because the data point in that direction).

ONE- OR TWO-SIDED?

Null hypothesis H_0 usually of the form: parameter=value (e.g. $\mu = 0.86$).

Alternative hypothesis H_a usually one of 3 types:

- **one-sided upwards**: parameter > value (e.g., $\mu > 0.86$),
- **one-sided downwards**: parameter < value (e.g., $\mu < 0.86$),
- **two-sided**: parameter different from value (e.g., $\mu \neq 0.86$).

Choice of alternative hypothesis:

- **one-sided**: when **focus is on particular alternative** (because other direction is difficult to interpret, or in beforehand of no interest),
- **two-sided**: **most common**, when no particular alternative is in focus or no knowledge is present in beforehand.

○ **affects the calculation of P -values**:

* **in general**, P -value is the probability of extreme events for H_0 relative to (i.e., in the direction of) H_a ,

* **example**: testing for population mean,

$H_0: \mu = \mu_0$:

$$H_a : \mu > \mu_0 : P = P(Z \geq z),$$

$$H_a : \mu < \mu_0 : P = P(Z \leq z),$$

$$H_a : \mu \neq \mu_0 : P = P(Z \geq |z|) + P(Z \leq -|z|).$$

* P -values and tests may also be termed one/two-sided.¹⁸

¹⁸ My recommendation is to only talk about one/two-sided alternative hypotheses.

TESTING BY CONFIDENCE INTERVAL

Fact: A confidence interval (CI) for a parameter with confidence level $C = 1 - \alpha$ can be used for a significance test at level α for

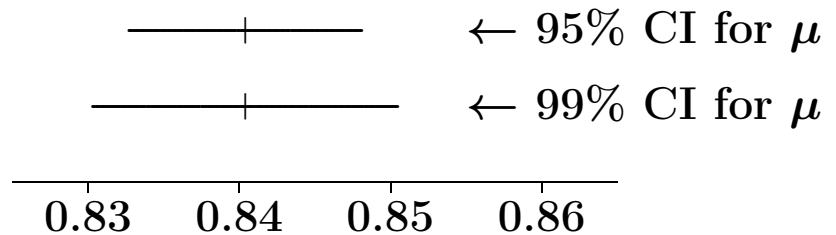
null hypothesis H_0 : parameter = value, versus
 alternative hypothesis H_a : parameter \neq value,

by the following “recipe”:

- reject H_0 , if value is outside interval.
- cannot reject H_0 , if value is inside interval,

Example II: Laboratory analysis,

- 95% CI for μ : $\bar{X} \pm 1.96 \sigma / \sqrt{3}$
 $= 0.8404 \pm 0.0077 \Rightarrow H_0: \mu = 0.86$,
 rejected at 5% level (also $H_0: \mu = 0.85$),



- 99% CI for μ : $\bar{X} \pm 2.576 \sigma / \sqrt{3}$
 $= 0.8404 \pm 0.0101 \Rightarrow H_0: \mu = 0.86$,
 rejected at 1% level (not $H_0: \mu = 0.85$).

Advantages and disadvantages of testing by use of CI:

- + easy (when CI done), enhances CI interpretation,
- no P-value.

SUMMARY NOTES

Key words and concepts:

- parameter, estimate, population,
- distribution of estimate/statistic, variability, standard error, bias,
- sample mean and proportion as unbiased estimates,
- law of large numbers (LLN), central limit theorem (CLT),
- **confidence interval**:
 - * concepts: confidence level, margin of error, frequency interpretation,
 - * z -formula for sample mean in normal distribution model (with known standard deviation σ).
- **statistical test**:
 - * concepts: null hypothesis H_0 , alternative hypothesis H_a (one or two-sided), test statistic and its (reference) distribution, P -value, significance level,
 - * possible conclusions: reject H_0 (and favour H_a), or no (insufficient) evidence against H_0 ,
 - * z -test formula for mean in normal distribution model (with known σ),
- relation between **test** and **confidence interval** (for a single parameter).

SUMMARY NOTES: FOUR-STEP PROCESSES

Four-step process for **confidence intervals** (PSLS 4e):

State: What is the practical question that requires estimating a parameter?

Plan: Identify a parameter and choose a level of confidence.

Solve: Carry out the work in two phases:

- * Check the conditions for the interval you plan to use.
- * Calculate the confidence interval (possibly using software).

Conclude: Return to the practical question to describe your results in this setting.

Four-step process for **tests** (PSLS 4e):

State: What is the practical question that requires a statistical test?

Plan: Identify a parameter, state the null and alternative hypotheses, and choose the type of test that fits your situation.

Solve: Carry out the test in three phases:

- * Check the conditions for the test you plan to use.
- * Calculate the test statistic.
- * Find the P -value using a table of Normal probabilities or technology.

Conclude: Return to the practical question to describe your results in this setting.

APPENDIX: NORMAL APPROXIMATION OF BINOMIAL DISTRIBUTION

For a **binomial distribution** (n, p) ($X \sim B(n, p)$) we have the approximations: ¹⁹

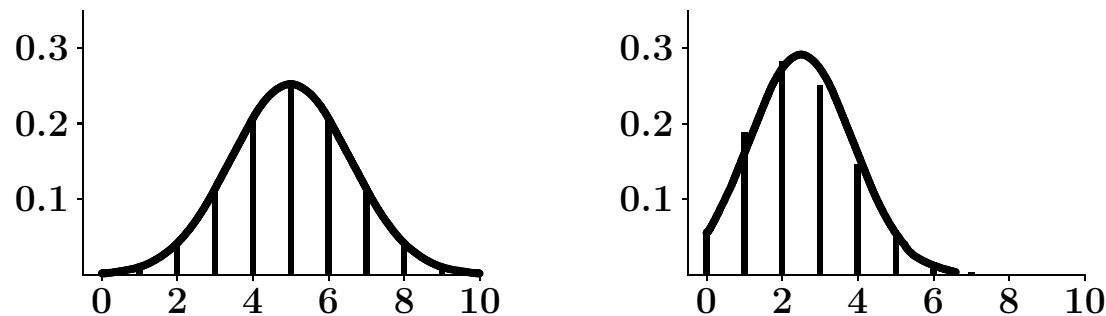
- $B(n, p) \approx N(np, \sqrt{np(1-p)})$,
- approximation “good” when $np(1-p) > 10$,²⁰
- formulas with **continuity correction** (± 0.5), for numbers $0 \leq x \leq n$ and $Z \sim N(0,1)$:

$$P(X \leq x) \approx P\left(Z \leq \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right),$$

$$P(X < x) \approx P\left(Z \leq \frac{x - 0.5 - np}{\sqrt{np(1-p)}}\right),$$

$$P(a \leq X \leq b) \approx P\left(Z \leq \frac{b+0.5-np}{\sqrt{np(1-p)}}\right) - P\left(Z \leq \frac{a-0.5-np}{\sqrt{np(1-p)}}\right),$$

Illustration for binomial distribution $B(10, p)$, with $p = 0.5$ (left) and $p = 0.25$ (right):



¹⁹ Illustrated by PSLS applet: Normal Approximation to Binomial Distributions.

²⁰ IPS 7e gives the slightly less strict rule: $np > 10$ and $n(1-p) > 10$. The PSLS/S texts have no specific rules, nor include the formula with continuity correction.